RELATIONS, FUNCTIONS AND BINARY OPERATIONS

(I) RELATIONS

TYPES OF REALTION

1. Empty Relation: A relation R on a set A is said to an empty relation iff \( R = \emptyset \) i.e. \((a, b) \in R \) \( \forall a, b \in A \) i.e. no element of A will be related to any other element of A with the help of the relation R.

2. Universal Relation: A relation R on a set A is said to be a universal relation iff \( R = A \times A \) i.e. \((a, b) \in R, \) \( \forall a, b \in A \) i.e. each element of A is related to every other element of A with the help of the relation R.

3. Reflexive Relation: A relation R on a set A is said to be a reflexive relation iff \((a, a) \in R \) \( \forall a \in A \) i.e. \( aRa \) \( \forall a \in A \)

4. Symmetric Relation: A relation R on a set A is said to be a symmetric relation iff \((a, b) \in R \Rightarrow (b, a) \in R \) i.e. if \((a, b) \in R \) then \((b, a) \in R \).

5. Transitive Relation: A relation R on a set A is said to be a transitive relation if \((a, b) \in R \) and \((b, c) \in R \) then \((a, c) \in R \) i.e. \((a, b),(b, c) \in R \Rightarrow (a, c) \in R \).

6. Equivalence Relation: A relation R on a set A is said to be an equivalence relation iff R is reflexive, symmetric, and transitive relation.

NOTE:

1. An equivalence relation on a set ‘A’ partition the set into mutually disjoint subsets each of which is an equivalence class.
2. If R is an equivalence relation on a set A and \( a \in A \) then the equivalence class of ‘a’ is denoted as \([a]\) and given by 
   \[ [a] = \{ x \in A : aRx \} \]
3. If \([a]\) is an equivalence class of a set A and \( b \in [a] \) then \([a] = [b] \).
4. In order to disprove i.e. to prove that a general statement does not hold or is not true in general we need to create an example.
5. On the other hand in order to establish a result in general we need to give its general proof.

FUNCTIONS

TYPES OF FUNCTIONS

1. ONE –ONE FUNCTION: A function \( y = f(x) \) is said to be one – one iff different pre-images have different images or if images are same then the pre-images are also same. i.e. \( f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \) or \( x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2) \)

Example: If \( f : R \rightarrow R \) is function given by

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Then the above functions are one-one.

2. MANY-ONE FUNCTION: A function in which atleast two pre-images have same image is called as many-one function.
Example: If \( f : \mathbb{R} \to \mathbb{R} \) is function given by
(i) \( f(x) = |x| \)
(ii) \( f(x) = x^2 \)
(iii) \( f(x) = \lfloor x \rfloor = \text{greatest integer smaller than equal to} \ 'x' \).
(iv) \( f(x) = k \), where ‘k’ is any constant.
(v) \( f(x) = x^4 + x^2 + 1 \)
Etc. are all many-one functions and one can verify the points in the domain where more than one point have same image.

3. INTO FUNCTION: A Function is said to be an ‘into’ function if there is atleast one element in the co-domain of the function such that it has no pre-image in domain.
Alternately, A function ‘f’ will be called an into function iff
\[ R_f \subset \text{codomain}(f) \]
Example: Following are the examples of ‘INTO’ functions:
(i) \( f : \mathbb{R} \to \mathbb{R} \) given by \( f(x) = e^x \).
(ii) \( f : \mathbb{R} \to \mathbb{R} \) given by \( f(x) = x^2 \).
(iii) \( f : [0, \infty) \to \mathbb{R} \) given by \( f(x) = \sqrt{x} \)
(iv) \( f : \mathbb{R} \to \mathbb{R} \) given by \( f(x) = \sin x \).

4. ONTO FUNCTIONS: A function is said to be an onto function if each element of co-domain has a pre-image in its domain.
Alternately, A function ‘f’ will be called an onto function iff
\[ R_f = \text{codomain}(f) \]
Examples
(i) \( f : \mathbb{R}^+ \to \mathbb{R} \) given by \( f(x) = \log x \).
(ii) \( f : \mathbb{R} \to \mathbb{R} \) given by \( f(x) = 2x + 3 \).
(iii) \( f : \mathbb{R} \to [-1,1] \) given by \( f(x) = \sin x \).
(iv) \( f : \mathbb{R} \to \mathbb{R} - \{0\} \) given by \( f(x) = \frac{1}{x} \).
(v) \( f : \mathbb{R} \to [0, \infty) \) given by \( f(x) = x^2 \).
5. INJECTIVE FUNCTION: A function which is one-one is also called as into function.
6. SURJECTIVE FUNCTION: A function which is onto is called as surjective function.
7. BIJECTIVE FUNCTION: A function which is one – one and onto both is called as bijective function.

**COMPOSITION OF FUNCTIONS**

If \( f \) and \( g \) are two functions then their composition

(i) \( f \circ g \) is defined iff \( R_g \subseteq D_f \) and \( f(g(x)) = f(g(x)) \)

(ii) \( g \circ f \) is defined iff \( R_f \subseteq D_g \) and \( g(f(x)) = g(f(x)) \)

**EXAMPLE:** If \( f(x) = \sin x \) and \( g(x) = \sqrt{x} \) then

\[
\begin{align*}
\quad f(g(x)) &= f(g(x)) \\
&= f(\sqrt{x}) \\
&= \sin \sqrt{x}
\end{align*}
\]

\[
\begin{align*}
\quad g(f(x)) &= g(f(x)) \\
&= g(\sin x) \\
&= \sqrt{\sin x}
\end{align*}
\]

**NOTE:**
1. If \( f : A \to B \) and \( g : B \to C \) then \( g \circ f : A \to C \).
2. If \( f : A \to B \) and \( g : B \to A \) then \( f \circ g \) and \( g \circ f \) are both defined, where \( g \circ f : A \to A \) and \( f \circ g : B \to B \).
3. If \( f : A \to A \) and \( g : A \to A \) then \( f \circ g \) and \( g \circ f \) are both defined, where \( g \circ f : A \to A \) and \( f \circ g : A \to A \).

**IDENTITY FUNCTION:** A function ‘I’ on a set \( A \) is said to be an identity function iff \( I : A \to A \), \( I(x) = x \).

**NOTE:** An identity function ‘I’ on \( A \) is also denoted as \( I_A \).

i.e. \( I_A : A \to A \) and \( I_A(x) = x \).

**EQUAL FUNCTIONS**

A function \( f \) will be equal to another function \( g \) iff

(i) \( D_f = D_g \)

(ii) \( f(x) = g(x) \ \forall \ x \in D_f \) or \( D_g \)

**INVERTIBLE FUNCTIONS:**
A function \( f : A \rightarrow B \) is said to be an invertible function iff \( \exists \) another function \( g : B \rightarrow A \) such that \( f \circ g = I_B \) and \( g \circ f = I_A \). And we write \( g = f^{-1} \)

**NOTE:**
1. A function is invertible iff it is one-one and onto.
2. In the above definition \( f \) and \( g \) are both inverse of each other i.e. \( f^{-1} = g \) and \( g^{-1} = f \).
3. \( (f^{-1})^{-1} = f \).
4. \( (f \circ g)^{-1} = g^{-1} \circ f^{-1} \).
5. \( f \circ f^{-1} = f^{-1} \circ f = I \)

**BINARY OPERATIONS**

‘*’ on a set ‘A’ will be called a binary operation iff \( * : A \times A \rightarrow A \) is a function

**NOTE:**
(i) \( \{a, b\} \) is written as \( a \ast b \).
(ii) \( A \times A = \{ (a, b) : a, b \in A \} \)

**EXAMPLE:**
(i) \( * : R \times R \rightarrow R \) defined by \( a \ast b = a + b \) is a binary operation (why?)
(ii) \( * : R \times R \rightarrow R \) defined by \( a \ast b = a - b \) is a binary operation (why?)
(iii) \( * : R \times R \rightarrow R \) defined by \( a \ast b = a \div b \) is not a binary operation (why?)
(iv) \( * : R \times R \rightarrow R \) defined by \( a \ast b = ab \) is a binary operation (why?)
(v) \( * : A \times A \rightarrow A \), where \( A = R - \{0\} \), defined by \( a \ast b = a \div b \) is a binary operation (why?)
(vi) \( * : R \times R \rightarrow R \) defined by \( a \ast b = a^3 + b^3 \) is a binary operation (why?)
(vii) \( * : N \times N \rightarrow N \) defined by \( a \ast b = a - b \) is not a binary operation (why?)

**TYPES OF BINARY OPERATION**

1. **COMMUTATIVE BINARY OPERATION:** A binary operation ‘*’ on a set A is called a commutative binary operation iff \( a \ast b = b \ast a \) \( \forall a, b \in A \)
2. **ASSOCIATIVE BINARY OPERATION:** A binary operation ‘*’ on a set ‘A’ is said to be an associative binary operation iff \( a \ast (b \ast c) = (a \ast b) \ast c \) \( \forall a, b, c \in A \)

**EXAMPLE**
(i) \( * : R \times R \rightarrow R \) defined by \( a \ast b = a + b \) is a commutative and associative binary operation (why?)
(ii) \( * : R \times R \rightarrow R \) defined by \( a \ast b = a - b \) is neither a commutative nor an associative binary operation (why?)
(iii) \( * : R \times R \rightarrow R \) defined by \( a \ast b = ab \) is a commutative and associative binary operation (why?)
(iv) $*: A \times A \rightarrow A$, where $A = R - \{0\}$, defined by $a*b = a \div b$ is neither a commutative nor associative binary operation (why?)

(v) $*: R \times R \rightarrow R$ defined by $a*b = a^3 + b^3$ is a commutative and associative binary operation (why?)

IDENTITY ELEMENT OF A BINARY OPERATION

If $*' is a binary operation on a set $A$ then an element $e$ in $A$ will be an identity element of the set $A$ iff $a*e = e*a = a, \forall a \in A$

EXAMPLE:

(i) In the binary operation $*: R \times R \rightarrow R$ defined by $a*b = a + b$, the number ‘0’ is an identity element.

(ii) In the binary operation $*: R \times R \rightarrow R$ defined by $a*b = ab$, the number ‘1’ is an identity element.

INVERSE OF AN ELEMENT IN A BINARY OPERATION

If $*' is a binary operation on a set $A$ then an element $b$ in $A$ will be inverse of the element $a$ iff $a*b = b*a = e$, where $e$ is an identity element of the binary operation.

EXAMPLE

(i) In the binary operation $*: R \times R \rightarrow R$ defined by $a*b = a + b$, the inverse of an element ‘a’ is ‘-a’. (Because: $a + (-a) = 0$)

(ii) In the binary operation $*: R \times R \rightarrow R$ defined by $a*b = ab$, the inverse of an element ‘a’ is $\frac{1}{a}$. (Because: $a \cdot \frac{1}{a} = 1$)

NOTE: If in a binary operation the identity element does not exist then their will be inverse of no element in the set.